Closing Wed night: HW_1A, 1B, 1C
Check out the first newsletter for homework hints, review sheets and old exam practice.

4.9 Antiderivatives

Def'n: If g(x) = f'(x), then we say

g(x) = "the derivative of f(x)", and

f(x) = "an antiderivative of g(x)"

Example:

Give an antiderivative of

$$g(x) = x^3 + 5$$

NOTE

$$x^n \to \frac{1}{n+1} x^{n+1}$$

$G(x) = \frac{1}{4} x^4 + 5x$

IS ONE ANSWER.

$G(x) = \frac{1}{4} x^4 + 5x + 37$

IS ANOTHER.

$G(x) = \frac{1}{4} x^4 + 5x - 141.2$

IS ANOTHER...

ALL ANSWERS WILL LOOK LIKE

$\boxed{G(x) = \frac{1}{4} x^4 + 5x + C \text{ WHERE } C = \text{a constant}}$

WE CALL THIS THE <u>GENERAL ANTIDERIVATIVE</u>

4.9: LIST OF GENERAL ANTIDERIVATIVES

FUNCTION	ANTIDERIVATIVE
$f(x) = x^n \ (n \neq -1)$	$F(x) = \dfrac{1}{n+1} x^{n+1} + C$
$f(x) = x^{-1} = \dfrac{1}{x}$	$F(x) = \ln\lvert x\rvert + C$
$f(x) = e^x$	$F(x) = e^x + C$
$f(x) = \cos(x)$	$F(x) = \sin(x) + C$
$f(x) = \sec^2(x)$	$F(x) = \tan(x) + C$
$f(x) = \sec(x)\tan(x)$	$F(x) = \sec(x) + C$
$f(x) = \sin(x)$	$F(x) = -\cos(x) + C$
$f(x) = \csc^2(x)$	$F(x) = -\cot(x) + C$
$f(x) = \csc(x)\cot(x)$	$F(x) = -\csc(x) + C$
$f(x) = \dfrac{1}{1+x^2}$	$F(x) = \tan^{-1}(x) + C$

Examples (you do):

Find the general antiderivative of

1. $f(x) = x^6 - 3$
2. $g(x) = \cos(x) + \frac{1}{x} + e^x + \frac{1}{1+x^2}$
3. $h(x) = \frac{5}{\sqrt{x}} + \sqrt[3]{x^2} \quad = 5x^{-1/2} + x^{3/2}$
4. $r(x) = \frac{x-3x^2}{x^3} \quad = \frac{x}{x^3} - \frac{3x^2}{x^3} = x^{-2} - 3\frac{1}{x}$

$\boxed{1}$ $F(x) = \frac{1}{7}x^7 - 3x + C$

$\boxed{2}$ $G(x) = \sin(x) + \ln|x| + e^x + \tan^{-1}(x) + C$

$\boxed{3}$ $H(x) = 5\left(\frac{1}{1/2}x^{1/2}\right) + \frac{1}{5/2}x^{5/2} + C = 10\sqrt{x} + \frac{2}{5}x^{5/2} + C$

$\boxed{4}$ $R(x) = \frac{1}{-1}x^{-1} - 3\ln|x| + C = -\frac{1}{x} - 3\ln|x| + C$

Initial Conditions: There is no way to know what "C" is unless we are given additional information about the antiderivative. Such information is called an **initial condition**.

Example: $f'(x) = e^x + 4x$ and
$f(0) = 5$

Find $f(x)$.

$\boxed{\text{STEP 1}}$ GENERAL ANTIDERIVATIVE

$$f(x) = e^x + 4\,\tfrac{1}{2}x^2 + C$$
$$= e^x + 2x^2 + C$$

$\boxed{\text{STEP 2}}$ INITIAL CONDITION

$$f(0) = 5 \Rightarrow e^{(0)} + 2(0)^2 + C = 5$$
$$1 + C = 5$$
$$\Rightarrow C = 4$$

$$\boxed{f(x) = e^x + 2x^2 + 4}$$

Example: $f''(x) = 15\sqrt{x}$, and

$$f(1) = 0, f(4) = 1$$

Find $f(x)$.

$$f'(x) = 15 \cdot \frac{2}{3} x^{3/2} + C$$

$$\boxed{f'(x) = 10x^{3/2} + C}$$

$$f(x) = 10 \cdot \frac{2}{5} x^{5/2} + Cx + D$$

$$f(x) = 4x^{5/2} + Cx + D$$

$f(1) = 0 \Rightarrow \quad 4 + C + D = 0 \Rightarrow C + D = -4 \Rightarrow D = -4 - C$

$f(4) = 1 \Rightarrow \quad 4(4)^{5/2} + C(4) + D = 1 \Rightarrow 4C + D = -127$

COMBINE AND SOLVE $\quad 4C + (-4 - C) = -127 \Rightarrow 3C = -123 \Rightarrow C = -41$

$D = -4 - -41$

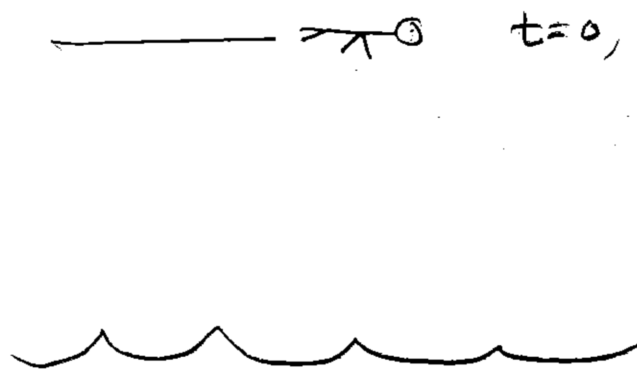
$D = 37$

$$\boxed{f(x) = 4x^{5/2} - 41x + 37}$$

CHECK!

Example:

Ron *steps off* the 10 meter high dive at his local pool. Find a formula for his height above the water. (Assume his acceleration is a constant 9.8 m/s^2 downward)



$t=0,\ h=10$

$$a(t) = -9.8 = h''(t)$$

$$v(t) = -9.8t + C = h'(t)$$

$$h(t) = -4.9t^2 + Ct + D = h(t)$$

"STEPS OFF" $\Rightarrow v(0)=0 \Rightarrow -9.8(0)+C=0 \Rightarrow C=0$

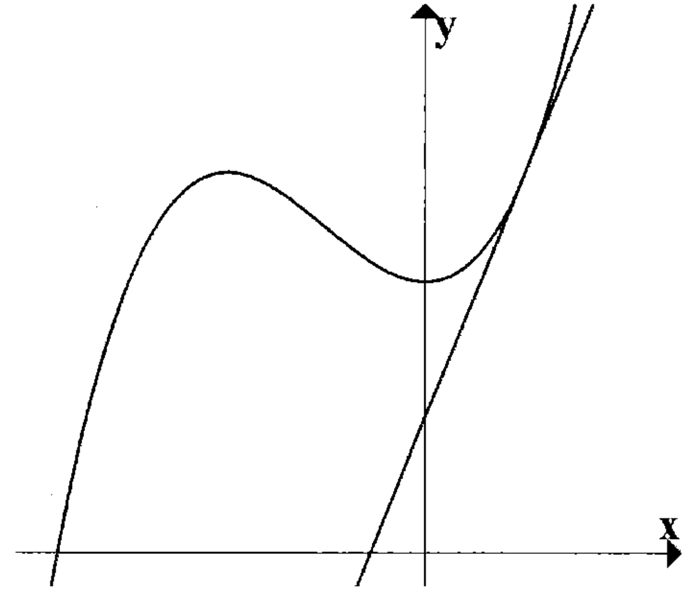
$h(0)=10 \Rightarrow -4.9(0)^2+(0)(0)+D=10 \Rightarrow D=10$

$$\boxed{h(t) = -4.9t^2+10}$$

5.1 Defining Area

Calculus is based on limiting processes that "approach" the exact answer to a rate question.

Calc. I Visual:



In Calculus I, you defined

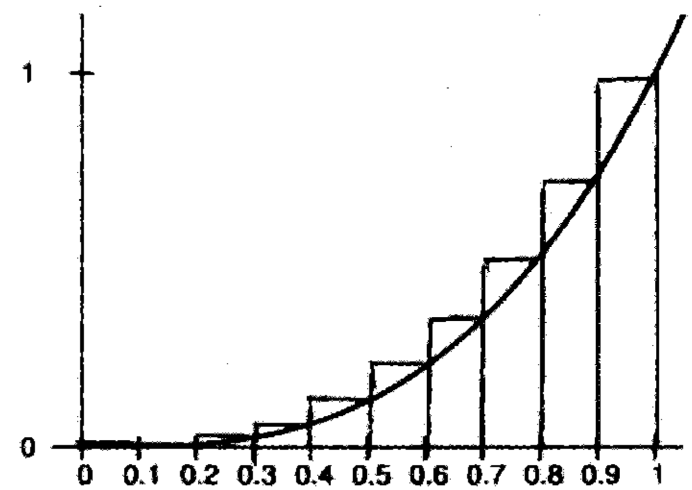
f′(x) = `slope of the tangent at *x*'

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

In Calculus II, we will see that antiderivatives are related to the area `under' a graph

$$= \lim_{n \to \infty} \sum_{i=1}^{n} f({x_i}^*)\Delta x$$

Calc. II Visual:



$R_{10} = 0.3025$

Riemann sums set up:

We are going to build a procedure to get better and better approximations of the area "under" $f(x)$.

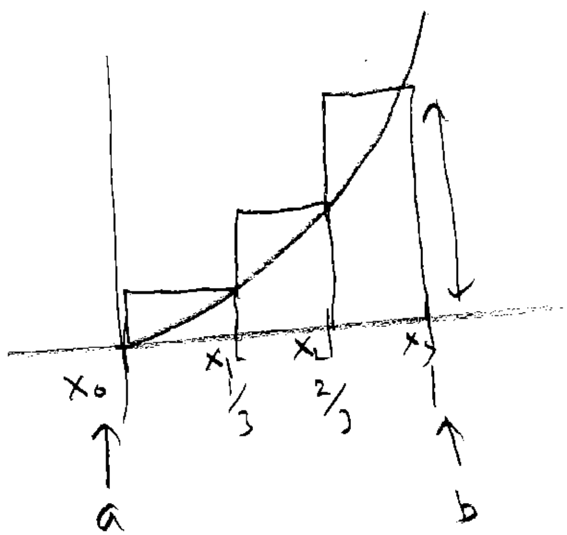
1. Break into *n* equal subintervals.

$$\Delta x = \frac{b-a}{n} \text{ and } x_i = a + i\Delta x$$

2. Draw *n* rectangles; use function.
   Area of each rectangle =

$$(\text{height})(\text{width}) = f(x_i^*)\Delta x$$

3. Add up rectangle areas.



Example:

Approximate the area under $f(x) = x^3$ from x = 0 to x = 1 using n = 3 subdivisions and *right-endpoints* to find the height.

$a = 0$

$b = 1$ $\quad f(x) = x^3$

$n = 3$

$\boxed{I}$ $\Delta x = \frac{b-a}{n} = \frac{1-0}{3} = 1/3$

$x_0 = a = 0$

$x_1 = a + \Delta x = 0 + 1/3 = 1/3$

$x_2 = a + 2\Delta x = 0 + 2 \cdot 1/3 = 2/3$

$x_3 = a + 3\Delta x = 0 + 3 \cdot 1/3 = 1$

$\boxed{II}$

$f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x$

$f(\frac{1}{3})\frac{1}{3} + f(\frac{2}{3})\, 1/3 + f(1)\Delta x$

$(\frac{1}{3})^3 \frac{1}{3} + (\frac{2}{3})^3 \frac{1}{3} + (1)^3 \frac{1}{3}$

$\approx 0.493827 = R_3$

OVERESTIMATE!

I did this again with 100 subdivisions, then 1000, then 10000. Here is the summary of my findings:

n	R_n	L_n
4	0.390625	0.140625
5	0.36	0.16
10	0.3025	0.2025
100	0.255025	0.245025
1000	0.25050025	0.24950025
10000	0.2499500025	0.2500500025

YOU DO! COMPUTE R_4

Pattern:

$$\Delta x = \frac{1-0}{n} = \frac{1}{n}, \qquad x_i = 0 + i\frac{1}{n} = \frac{i}{n}$$

Pattern for each Rectangle

$$\text{Height} = f(x_i) = x_i^3 = \left(\frac{i}{n}\right)^3$$

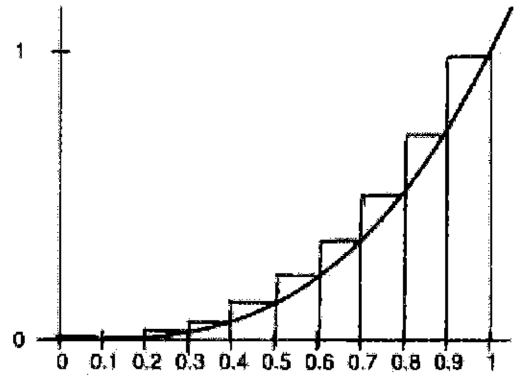
$$\text{Area} = f(x_i)\Delta x = x_i^3 \Delta x = \left(\frac{i}{n}\right)^3 \frac{1}{n}$$

Adding up the area of each rectangle

$$\text{Sum} = \sum_{i=1}^{n} x_i^3 \Delta x = \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3 \frac{1}{n}$$

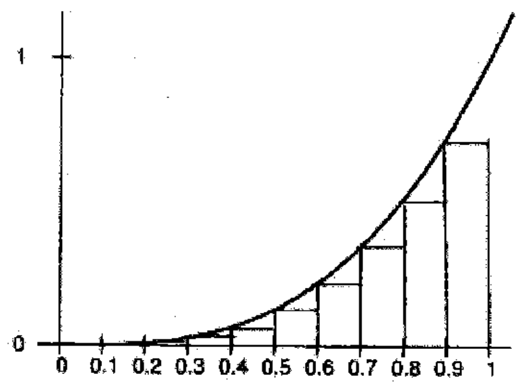
$$\text{Exact Area} = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3 \frac{1}{n}$$

Estimating the Area with 10 Rectangles and Right Endpoints



$$R_{10} = S_1 + S_2 + \cdots + S_9 + S_{10}$$
$$R_{10} = w_1 h_1 + w_2 h_2 + \cdots + w_9 h_9 + w_{10} h_{10}$$
$$R_{10} = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_9)\Delta x + f(x_{10})\Delta x$$
$$R_{10} = \sum_{i=1}^{10} f(x_i)\Delta x = \frac{1}{10}\left(\frac{1}{10}\right)^3 + \frac{1}{10}\left(\frac{2}{10}\right)^3 + \cdots + \frac{1}{10}\left(\frac{10}{10}\right)^3$$
$$R_{10} = 0.3025$$

Estimating the Area with 10 Rectangles and Left Endpoints



$$L_{10} = S_1 + S_2 + \cdots + S_9 + S_{10}$$
$$L_{10} = w_1 h_1 + w_2 h_2 + \cdots + w_9 h_9 + w_{10} h_{10}$$
$$L_{10} = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_8)\Delta x + f(x_9)\Delta x$$
$$L_{10} = \sum_{i=1}^{10} f(x_{i-1})\Delta x = \frac{1}{10}\left(\frac{0}{10}\right)^3 + \frac{1}{10}\left(\frac{1}{10}\right)^3 + \frac{1}{10}\left(\frac{2}{10}\right)^3 + \cdots + \frac{1}{10}\left(\frac{9}{10}\right)^3$$
$$L_{10} = 0.2025$$

Summary of Finding the Area under $y = x^3$ from 0 to 1

n	R_n	L_n
4	0.390625	0.140625
5	0.36	0.16
10	0.3025	0.2025
100	0.255025	0.245025
1000	0.25050025	0.24950025
10000	0.2499500025	0.2500500025

For our example, $f(x) = x^3$ and we are finding the area under the graph from $a = 0$ and $b = 1$. So we have

$$\Delta x = \frac{1-0}{n} = \frac{1}{n} \qquad \text{and} \qquad x_i = 0 + i\frac{1}{n} = \frac{i}{n}$$

$$\text{The exact area } = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3 \frac{1}{n}$$

In general, we can find the exact area under the graph of $f(x)$ from $x = a$ to $x = b$ by

$$\Delta x = \frac{b-a}{n} \qquad \text{and} \qquad x_i = a + i\Delta x$$

$$\text{The exact area } = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f(a + i\Delta x)\Delta x$$

Aside: Computing $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3 \frac{1}{n}$ exactly.

$$\lim_{n\to\infty} \left[\left(\frac{1}{n}\right)^3 \frac{1}{n} + \left(\frac{2}{n}\right)^3 \frac{1}{n} + \cdots + \left(\frac{n}{n}\right)^3 \frac{1}{n}\right]$$

$$= \lim_{n\to\infty} \left[\frac{1}{n^4}\left(\underbrace{1^3 + 2^3 + 3^3 + \cdots + n^3}_{= \left(\frac{n(n+1)}{2}\right)^2}\right)\right] \leftarrow \text{factoring out } \frac{1}{n^4} \text{ in each term}$$

$= \left(\frac{n(n+1)}{2}\right)^2 \leftarrow$ formula in book

$$= \lim_{n\to\infty} \left[\frac{1}{n^4}\left(\frac{n(n+1)}{2}\right)^2\right] \leftarrow \text{multiply this out and compute the limit. (It will be } 1/4 = 0.25)$$

Definition of the Definite Integral

We define the exact area "under" f(x) from x = a to x = b curve to be

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x,$$

where $\Delta x = \frac{b-a}{n}$ and

$$x_i = a + i\Delta x.$$

We call this the definite integral of f(x) from x = a to x = b, and we write

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

Example: Write down this definition for the function $f(x) = \sqrt{x}$ on the interval $x = 5$ to $x = 7$.

$a = 5$

$b = 7$

$f(x) = \sqrt{x}$

$$\Delta x = \frac{b-a}{n} = \frac{7-5}{n} = \frac{2}{n}$$

$$x_i = a + i\Delta x = 5 + i\frac{2}{n} = 5 + \frac{2i}{n}$$

$$f(x_i)\Delta x = \sqrt{x_i}\,\Delta x = \sqrt{5+\frac{2i}{n}}\,\frac{2}{n}$$

$$\int_5^7 \sqrt{x}\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{5+\frac{2i}{n}}\,\frac{2}{n}$$